

ON SCATTERING CONSTANTS FOR A NON-CONGRUENCE SUBGROUP

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ABSTRACT. Scattering constants are special values of Dirichlet series associated to non-holomorphic Eisenstein series. In this paper we give closed formulas for the scattering constants related to a non-congruence subgroup obtained via a Belyi map of an elliptic curve.

INTRODUCTION

Let $\Gamma \subset SL_2(\mathbb{Z})$ be a finite index subgroup. To every cusp S_j of Γ we can associate a non-holomorphic Eisenstein series $E_j^\Gamma(z, s)$ and for two cusps S_j and S_k the scattering constant C_{jk}^Γ is defined by the constant term of a Dirichlet series coming from the Fourier expansion of $E_j^\Gamma(z, s)$ in the cusp S_k .

Given an algebraic curve C and a Belyi map β , i.e. a morphism $\beta : C \rightarrow \mathbb{P}^1$ that is ramified in at most three points, for simplicity we assume that both are defined over \mathbb{Q} , then there exists a subgroup $\Gamma \subset \Gamma(2)$ such that $C(\mathbb{C}) \cong \overline{\Gamma \backslash \mathbb{H}}$. The cusps of Γ correspond to the ramification points of (C, β) . Arakelov theory gives us an expression for the Néron Tate height (see theorem 2.5 below for more details)

$$\text{ht}_{NT}(\text{ramification points}) = \text{weighted sum of } C_{jk}^\Gamma s + \text{algebraic term.}$$

Our main result concerns the elliptic curve $E : y^2 = x^3 + 5x + 10$ and the non-congruence subgroup Γ_E for the Belyi map $\beta_E : E \rightarrow \mathbb{P}^1$ given by $\beta_E(x, y) = \frac{y(x-5)+16}{32}$. This curve is referred to as 400H1 in Cremona's tables [Cr], its Mordell Weil group has rank one and is generated by $P = (1, 4)$. The zero element \mathcal{O} and P as well as $-P$ correspond to cusps of the group Γ_E .

Theorem: For the non-congruence subgroup Γ_E as above the scattering constants $C_{\mathcal{O}, \mathcal{O}}^{\Gamma_E}$, $C_{P, P}^{\Gamma_E}$ and $C_{P, -P}^{\Gamma_E}$ are given by

$$\begin{aligned} C_{\mathcal{O}, \mathcal{O}}^{\Gamma_E} &= \frac{1}{30} \left(C^{\Gamma(1)} - \frac{1}{\pi} (14 \log(2) + 6 \log(5)) \right) \\ C_{P, P}^{\Gamma_E} &= \frac{1}{120} \left(4C^{\Gamma(1)} + \frac{1}{\pi} (-131 \log(2) + 15 \log(5) - 60 \text{ht}_{NT}(P)) \right) \\ C_{P, -P}^{\Gamma_E} &= \frac{1}{120} \left(4C^{\Gamma(1)} + \frac{1}{\pi} (-71 \log(2) + 60 \text{ht}_{NT}(P)) \right), \end{aligned}$$

where

$$(0.1) \quad C^{\Gamma(1)} = -\frac{6}{\pi} (12\zeta'(-1) - 1 + \log(4\pi))$$

is the unique scattering constant for the full modular group $\Gamma(1)$.

All other scattering constants for Γ_E are \mathbb{Q} -linear combinations of the four scattering constants given above. The coefficients depend on the ramification data coming from β_E .

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Remark: Previous to these explicit expressions, nearly no formulas for scattering constants have been available. For Eisenstein series coming from certain congruence subgroups of $SL_2(\mathbb{Z})$ formulas for the scattering matrices are known ([He], [Hu]). Beside that, Venkov studied cycloidal groups [Ve]. From these results scattering constants can be deduced.

1. EISENSTEIN SERIES

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ be the upper half plane and $\Gamma(1) = PSL_2(\mathbb{Z})$. Then $\Gamma(1)$ acts on \mathbb{H} by the Möbius transformation. These action can be extended to $\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{Q} \cup \infty$. Let $\Gamma \subset \Gamma(1)$ be a finite index subgroup then $\overline{\mathbb{Q}} = \mathbb{Q} \cup \infty$ is divided into finitely many equivalence classes with respect to the action of Γ ; the classes are called cusps of Γ . We will use the word cusp as well for a representative of a cusp. Let $S_j \in \overline{\mathbb{Q}}$ be a cusp and Γ_j its stabilizer in Γ . For S_j there is a $\gamma_j \in \Gamma(1)$ with $\gamma_j(\infty) = S_j$ and a $b_j \in \mathbb{N}$, such that

$$\sigma_j^{-1} \Gamma_j \sigma_j = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle \quad \text{with } \sigma_j = \gamma_j \cdot \begin{pmatrix} \sqrt{b_j} & 0 \\ 0 & 1/\sqrt{b_j} \end{pmatrix}.$$

The number b_j is called the width of the cusp S_j .

Definition 1.1. Let $\Gamma \subset \Gamma(1)$ be a finite index subgroup. For each cusp S_j there is a non-holomorphic Eisenstein series $E_j^\Gamma(z, s)$, which for $z \in \mathbb{H}$, $s \in \mathbb{C}$ and $\text{Re } s > 1$ is defined by the convergent series

$$E_j^\Gamma(z, s) = \sum_{\sigma \in \Gamma_j \backslash \Gamma} \text{Im}(\sigma_j^{-1} \sigma(z))^s = b_j^{-s} \sum_{\sigma \in \Gamma_j \backslash \Gamma} \text{Im}(\gamma_j^{-1} \sigma(z))^s.$$

Properties: Let us recall some facts on the theory of Eisenstein series; the standard reference is [Ku]. The function $E_j^\Gamma(z, s)$ has a meromorphic continuation to the s -plane, with a simple pole in $s = 1$ with residue $3/(\pi \cdot [\Gamma(1) : \Gamma])$. For all $\gamma \in \Gamma$ we have $E_j^\Gamma(\gamma(z), s) = E_j^\Gamma(z, s)$. The Fourier expansion of $E_j^\Gamma(z, s)$ at the cusp S_k is given by

$$E_j^\Gamma(\sigma_k(z), s) = \delta_{jk} \cdot y^s + \pi^{1/2} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \cdot \phi_{jk}^\Gamma(s) \cdot y^{s-1} + \sum_{m \neq 0} a_m(y, s) e^{2\pi i m x}$$

where $z = x + iy$ and $\Gamma(s)$ is the Gamma function. Furthermore we have

$$(1.1) \quad \phi_{jk}^\Gamma(s) = \frac{1}{(b_j b_k)^s} \sum_{c > 0} r_{jk}^\Gamma(c) \frac{1}{c^{2s}}$$

and

$$(1.2) \quad r_{jk}^\Gamma(c) = \# \{d \pmod{b_k c} \mid \exists \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \gamma_j^{-1} \Gamma \gamma_k\}.$$

Then the *scattering matrix*

$$\Phi_\Gamma(s) = \left(\pi^{1/2} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \cdot \phi_{jk}^\Gamma(s) \right)_{j,k}$$

is symmetric. Note that all the coefficients of the scattering matrix are Dirichlet series in a general sense. They have a meromorphic continuation with a simple pole in $s = 1$ of residue $3/(\pi \cdot [\Gamma(1) : \Gamma])$.

Definition 1.2. For all pairs j, k we define the scattering constant C_{jk}^Γ to be the constant term at 1 of the Dirichlet series $(\Phi_\Gamma)_{jk}(s)$, i.e.

$$(1.3) \quad C_{jk}^\Gamma := \lim_{s \rightarrow 1} \left((\Phi_\Gamma(s))_{j,k} - \frac{3/(\pi \cdot [\Gamma(1) : \Gamma])}{s - 1} \right).$$

We will need the values of the scattering constants for $\Gamma(1)$ and $\Gamma(2)$. The group $\Gamma(1)$ has one cusp, hence one scattering constant and it has already been introduced in (0.1). For the group $\Gamma(2)$ there exist only two different scattering constants, although $\Gamma(2)$ has three cusps. These constants are

$$(1.4) \quad \begin{aligned} C_a^{\Gamma(2)} &= -\frac{1}{3\pi} (36\zeta'(-1) - 3 + 3\log(4\pi) + 7\log(2)) \\ &= \frac{1}{6}C^{\Gamma(1)} - \frac{7}{3\pi}\log(2) \end{aligned}$$

$$(1.5) \quad \begin{aligned} C_b^{\Gamma(2)} &= -\frac{1}{3\pi} (36\zeta'(-1) - 3 + 3\log(4\pi) + \log(2)) \\ &= \frac{1}{6}C^{\Gamma(1)} - \frac{1}{3\pi}\log(2), \end{aligned}$$

where the first case is the one with $S_j = S_k$ and in the second case we have $S_j \neq S_k$. For a calculation see e.g. [Po].

The scattering constants for a group can be constructed from the constants for a subgroup. Conversely, the knowledge of scattering constants for a group gives us some information about sums of scattering constants for subgroups. Take the group $\Gamma(2)$, then we get:

Proposition 1.3. *Let Γ be a finite index subgroup of $\Gamma(2)$, $S_j^{\Gamma(2)}$ and $S_{k'}^{\Gamma(2)}$ two cusps of $\Gamma(2)$. Then*

$$(1.6) \quad \sum_{S_i^{\Gamma} \subset S_j^{\Gamma(2)}} \frac{b_i}{2} C_{ik}^{\Gamma} = C_{jk'}^{\Gamma(2)} - \frac{1}{2\pi[\Gamma(2) : \Gamma]} \sum_{S_i^{\Gamma} \subset S_j^{\Gamma(2)}} \frac{b_i}{2} \log\left(\frac{b_i b_k}{4}\right),$$

where we sum over a system of representatives $\{S_i^{\Gamma}\}$ of cusps of Γ such that $S_i^{\Gamma} \sim_{\Gamma(2)} S_j^{\Gamma(2)}$ and S_k^{Γ} is any cusp of Γ with $S_k^{\Gamma} \sim_{\Gamma(2)} S_{k'}^{\Gamma(2)}$. The b_i and b_k denote the widths of the cusps.

Proof: The formula is a consequence of the relation of Eisenstein series

$$\begin{aligned} 2^s E_j^{\Gamma(2)}(z, s) &= \sum_{\gamma \in \Gamma_j \backslash \Gamma(2)} \text{Im}(\gamma_j \gamma(z))^s = \sum_{-\frac{d}{c} \in S_j^{\Gamma(2)}} \frac{\text{Im}(z)^s}{|cz - d|^{2s}} \\ &= \sum_{S_i^{\Gamma} \subset S_j^{\Gamma(2)}} \sum_{-\frac{d}{c} \in S_i^{\Gamma}} \frac{\text{Im}(z)^s}{|cz - d|^{2s}} \\ &= \sum_{S_i^{\Gamma} \subset S_j^{\Gamma(2)}} b_i^s E_i^{\Gamma}(z, s). \end{aligned}$$

From the implied relation for the constant terms of the Fourier expansions we conclude the claim by a straight forward calculation. □

2. BELYI'S THEOREM, NÉRON TATE HEIGHTS AND ARAKELOV THEORY

In 1979 G. Belyi proved in [Be]

Theorem 2.1. *Let C be a non-singular algebraic curve defined over a number field. Then there exists a finite morphism $\beta : C \rightarrow \mathbb{P}^1$ with at most the three critical values $0, 1, \infty$.*

This was the missing step for the following equivalences.

Theorem 2.2. *Let C be a non-singular algebraic curve over \mathbb{C} . Then the following are equivalent*

- (i) *The curve C is defined over a number field.*

- (ii) *There exists a finite morphism $\beta_C : C \longrightarrow \mathbb{P}^1$ with at most the three critical values $0, 1, \infty$.*
- (iii) *There is a subgroup $\Gamma_C \subset \Gamma(2)$ such that $C(\mathbb{C}) \cong \overline{\Gamma_C \setminus \mathbb{H}}$.*

Proof: See e.g. [Bi], [Bo] or [Se]. □

Definition 2.3. *A pair consisting of a curve and a map with the properties from (ii) is called Belyi pair. The map alone is a Belyi map.*

Some data from the Belyi pair have a direct counterpart in the group Γ_C . There is a 1-1 correspondence between the cusps of Γ_C and the ramification points of the Belyi pair. The widths of the cusps resemble the ramification orders.

Definition 2.4. *We will call the ramification points of a Belyi pair cusps. A divisor on C which has only support in the cusps of (C, β_C) is called cuspidal divisor.*

Belyi pairs allow to formulate the following theorem concerning Néron Tate heights.

Theorem 2.5. *Let $\beta : K \rightarrow \mathbb{P}^1$ be a Belyi map for an algebraic curve C over \mathbb{Q} with induced Belyi uniformization $C(\mathbb{C}) \cong \overline{\Gamma_C \setminus \mathbb{H}}$. Let $D = \sum_j n_j S_j$ and $D' = \sum_k m_k S_k$ be two cuspidal divisors of degree 0. Then the Néron Tate height pairing of D and D' is given by:*

$$(2.1) \quad \langle D, D' \rangle_{NT} = - \sum_{p \text{ prim}} \delta_p \log(p) - 2\pi \sum_{j,k} n_j m_k C_{jk}^{\Gamma_C}.$$

The coefficients δ_p are rational numbers, that are explicitly computable (see (2.2)) and $C_{jk}^{\Gamma_C}$ is the scattering constant for the cusps S_j and S_k from Γ_C .

Proof: See [Kü]. □

The above coefficients δ_p , as we will explain now, are given via local intersection numbers.

Let C/\mathbb{Q} be an algebraic curve and \mathcal{C}/\mathbb{Z} be a proper regular model. Let $\mathbf{D}_1, \mathbf{D}_2 \in \text{Div}(\mathcal{C})$ be two prime divisors with no common components and $x \in \mathbf{D}_1 \cap \mathbf{D}_2$. Fix local parameters f_1, f_2 for \mathbf{D}_1 and \mathbf{D}_2 . We define the local intersection number at x to be

$$i_x(\mathbf{D}_1, \mathbf{D}_2) = \ell_{\mathcal{O}_{\mathcal{C},x}}(\mathcal{O}_{\mathcal{C},x}) / (f_1, f_2).$$

Further, we define the total local intersection number of \mathbf{D}_1 and \mathbf{D}_2 at a prime p to be

$$i_p(\mathbf{D}_1, \mathbf{D}_2) = \sum_{x \in \mathbf{D}_1 \cap \mathbf{D}_2 \cap \mathcal{C}_p} i_x(\mathbf{D}_1, \mathbf{D}_2) [k_x : k_p].$$

For two divisors $\mathbf{D} = \sum n_j \mathbf{D}_j$, $\mathbf{D}' = \sum m_k \mathbf{D}'_k$ with no common components we define by linearity

$$i_p(\mathbf{D}, \mathbf{D}') = \sum_{j,k} n_j m_k i_p(\mathbf{D}_j, \mathbf{D}'_k).$$

Now we are looking at the different fibers of the scheme \mathcal{C}/\mathbb{Z} at once.

Definition 2.6. *For two divisors $\mathbf{D}, \mathbf{D}' \in \text{Div}(\mathcal{C})$ with no common component we define the intersection number at the finite places by*

$$(\mathbf{D}, \mathbf{D}')_{fin} = \sum_p i_p(\mathbf{D}, \mathbf{D}') \log(p).$$

From now on \mathbf{D} denotes the Zariski closure of a divisor D on C . The group of divisors on C with degree zero will be denoted by $\text{Div}_0(C)$ and $\text{Div}_p(\mathcal{C})$ is the set of all divisors supported on \mathcal{C}_p , here $\mathcal{C}_p = \mathcal{C} \times k_p$ ($k_p = \mathbb{Z}/p\mathbb{Z}$) denotes the special fiber of \mathcal{C} at the place p .

Lemma 2.7. *There exists a unique linear map*

$$\Phi_p : \text{Div}_0(C) \rightarrow \mathbb{Q} \otimes \text{Div}_p(\mathcal{C}) / (\mathbb{Q} \otimes \mathcal{C}_p),$$

such that for all $D \in \text{Div}_0(C)$ the divisor $\mathbf{D} + \Phi_p(D)$ is orthogonal to $\text{Div}_p(\mathcal{C})$.

Proof: See e.g. [Hr]. □

The correction divisor Φ for $D \in \text{Div}_0(C)$ is defined by

$$\Phi(D) = \sum_p \Phi_p(D).$$

Definition 2.8. *We define for two divisors $D, D' \in \text{Div}_0(C)$ with no common components*

$$(D, D')_{fin} = (\mathbf{D} + \Phi(D), \mathbf{D}' + \Phi(D'))_{fin}.$$

From the definitions we see

$$(2.2) \quad (D, D')_{fin} = \sum_p \delta_p \log(p).$$

These δ_p are exactly the δ_p of theorem 2.5, whenever D and D' have no common components. Since the Néron Tate height pairing vanishes on divisors of rational functions, we may reduce the general case to the above situation by replacing D' with $D' + \text{div}(f)$ for a suitable rational function f on C .

Remark 2.9. *As a consequence of the functoriality of the intersection number at the finite places w.r.t. pull-back morphisms (see [La]), the quantity $(D, D')_{fin}$ does not depend on the particular chosen regular model.*

3. AN ELLIPTIC BELYI PAIR AND ITS SCATTERING CONSTANTS

Now, we will focus on one particular Belyi pair considered by Elkies in [El1].

Proposition 3.1. *The elliptic curve*

$$(3.1) \quad E : y^2 = x^3 + 5x + 10$$

together with the map

$$(3.2) \quad \begin{aligned} \beta : E &\longrightarrow \mathbb{P}^1 \\ (x, y) &\longmapsto \frac{y(x-5) + 16}{32} \end{aligned}$$

form a Belyi pair.

Proof: Regard the map $(x, y) \mapsto y(x-5)$. It is easy to check, that the critical values are $\{\infty, \pm 16\}$. The map (3.2) above normalizes the critical values to 0, 1 and ∞ . □

Proposition 3.2. *The Mordell Weil group $E(\mathbb{Q})$ of the elliptic curve defined via (3.1) is of rank one with trivial torsion. The point $S_1 = (1, 4)$ generates $E(\mathbb{Q})$. The group Γ_E associated with the Belyi pair (E, β_E) is a non-congruence subgroup.*

Proof: For the rank, torsion, and the generator look at Cremona's tables in [Cr]. The curve is listed under 400H1. The point S_1 generates an infinite group. Hence, the cuspidal divisor $S_1 - \mathcal{O}$ is not torsion in the Picard group. The theorem of Manin and Drinfeld [Dr], [Ma] (see also [El2]) then shows that Γ_E is non-congruence. □

Remark 3.3. *It was shown in [Po] that $\Gamma_E = \varphi^{-1}(\text{Stab}_{S_5}(5))$ with $\varphi : \Gamma(2) \rightarrow S_5$ given by the images of the generators of $\Gamma(2)$: $\varphi(\gamma_0) = (1235)$ and $\varphi(\gamma_1) = (1234)$. But such a description of Γ_E is not needed to achieve the results of this paper.*

The ramification points, i.e. the cusps of the Belyi pair, are

$$(3.3) \quad \begin{aligned} S_0 &= \mathcal{O} \\ S_1 &= (1, 4), \quad S_2 = -4S_1 = (6, -16), \\ S_3 &= -S_1 = (1, -4), \quad \text{and} \quad S_4 = 4S_1 = (6, 16), \end{aligned}$$

where \mathcal{O} corresponds to the point ∞ ; it is the neutral element of $E(\mathbb{Q})$. The point S_0 lies above ∞ , S_1 and S_2 above 0 and S_3 as well as S_4 above 1. The notation S_i with $i \in \{0, \dots, 4\}$ will be used for the divisors of the cusps in $\text{Div}(E)$ and also for the corresponding cusps of the group Γ_E .

The scattering constants $C_{jk}^{\Gamma_E}$ will be determined below via an application of theorem 2.5 on the curve E defined via (3.1) together with the additional relations coming from proposition 1.3.

In particular, we need certain intersection numbers at the finite places, these will be determined in section 4 using an appropriate model.

4. CALCULATION OF SOME COEFFICIENTS δ_p FOR E

The elliptic curve E defined via (3.1) has a minimal proper regular model over \mathbb{Z} . We denote the Zariski closure of S_j on the model with \mathbf{S}_j .

Theorem 4.1. *An illustration of the model and the behavior of the divisors associated to the points from (3.3) can be seen in figure 1.*

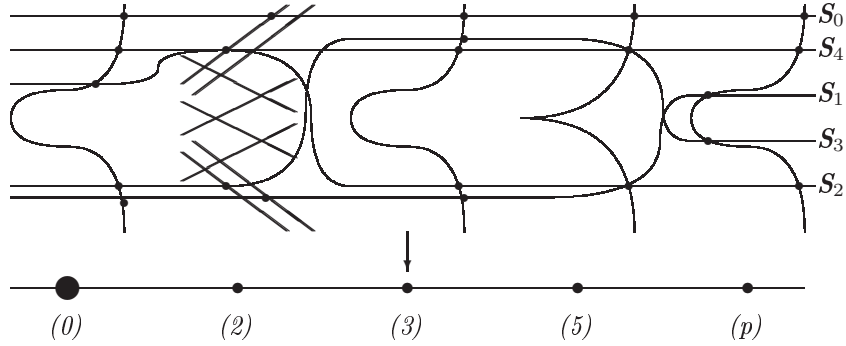


FIGURE 1. The associated divisors on the minimal proper regular model of the curve 400H1

The model fulfills the following properties:

- (i) All special fibers of \mathcal{E} consist of one irreducible component except the fiber above (2).
- (ii) The fiber above (2) consists of eight irreducible components with the following multiplicities

$$\mathcal{E}_2 = \mathcal{C}_1 + \mathcal{C}_2 + 2\mathcal{C}_3 + 2\mathcal{C}_4 + 2\mathcal{C}_5 + 2\mathcal{C}_6 + \mathcal{C}_7 + \mathcal{C}_8$$

and the intersection matrix of the fiber over (2) is given by table 1.

- (iii) There are only three points on the model where two of the divisors $\mathbf{S}_0, \dots, \mathbf{S}_4$ intersect. We denote these points by

$$x_{12} = \mathbf{S}_1 \cap \mathbf{S}_2 \quad x_{34} = \mathbf{S}_3 \cap \mathbf{S}_4 \quad x_{24} = \mathbf{S}_2 \cap \mathbf{S}_4.$$

(\cdot)	\mathcal{C}_1	\mathcal{C}_2	$2\mathcal{C}_3$	$2\mathcal{C}_4$	$2\mathcal{C}_5$	$2\mathcal{C}_6$	\mathcal{C}_7	\mathcal{C}_8
\mathcal{C}_1	-2	0	2	0	0	0	0	0
\mathcal{C}_2	0	-2	2	0	0	0	0	0
$2\mathcal{C}_3$	2	2	-8	4	0	0	0	0
$2\mathcal{C}_4$	0	0	4	-8	4	0	0	0
$2\mathcal{C}_5$	0	0	0	4	-8	4	0	0
$2\mathcal{C}_6$	0	0	0	0	4	-8	2	2
\mathcal{C}_7	0	0	0	0	0	2	-2	0
\mathcal{C}_8	0	0	0	0	0	2	0	-2

TABLE 1. Intersection matrix for \mathcal{E}_2

- (iv) *The divisors $\mathbf{S}_0, \dots, \mathbf{S}_4$ only intersect with one component of the special fiber above (2) and these intersection numbers are*

$$(\mathbf{S}_0, \mathcal{C}_1)_{fin} = (\mathbf{S}_1, \mathcal{C}_7)_{fin} = (\mathbf{S}_2, \mathcal{C}_1)_{fin} = (\mathbf{S}_3, \mathcal{C}_8)_{fin} = (\mathbf{S}_4, \mathcal{C}_1)_{fin} = 1$$

while all other intersection numbers vanish.

Proof: The Tate algorithm provides an explicit method to obtain the model and its properties (i), (ii). Keeping track of the points S_j and their induced divisors \mathbf{S}_j in all steps of these calculations gives (iii) and (iv); a detailed calculation is in [Bu]. \square

Now, we need to calculate two more ingredients to determine the possible δ_p 's for E : the local intersection numbers and the correction divisors.

Local intersection numbers: We will denote the local intersection number at x_{jk} with m_{jk} . Explicit calculations show, see e.g. [Bu], p. 79, that the local ring at the point x_{24} is given by

$$\mathcal{O}_{\mathcal{E}, x_{24}} = \mathbb{Z}[x, y]/(y^2 - x^3 - 5x - 10)_{((2, x, y))}$$

and the local equations of \mathbf{S}_2 and \mathbf{S}_4 are $f_2 = y - 16$ and $f_4 = y + 16$.

Thus, the module $\mathcal{O}_{\mathcal{E}, x_{24}}/(f_2, f_4)$ is given by

$$\begin{aligned} \mathcal{O}_{\mathcal{E}, x_{24}}/(f_2, f_4) &= (\mathbb{Z}[x, y]/(y^2 - x^3 - 5x - 10)_{((2, x, y))})/(y - 16, y + 16) \\ &= \mathbb{Z}[x, y]/(y^2 - x^3 - 5x - 10, y - 16, y + 16)_{((2, x, y))} \\ &= \mathbb{Z}[x]/(-x^3 - 5x + 246, 32)_{((2, x))} \\ &= \mathbb{Z}/(32)_{((2))}. \end{aligned}$$

The intersection number is therefore given by

$$m_{24} = i_{x_{24}}(S_2, S_4) = \ell_{\mathcal{O}_{\mathcal{E}, x_{24}}} \mathcal{O}_{\mathcal{E}, x_{24}}/(f_2, f_4) = \ell_{\mathcal{O}_{\mathcal{E}, x_{24}}} \mathbb{Z}/(32) = 5.$$

Similar calculations for the other two points give

$$m_{12} = m_{34} = 1.$$

Correction divisors: We will work with cuspidal divisors that are the difference of two cusps and we define $D_{jk} = S_j - S_k$, with $j, k \in \{0, \dots, 4\}$. The Zariski closure of D_{jk} will be denoted with \mathbf{D}_{jk} . Next we will calculate the divisors $\Phi_{jk} = \Phi(D_{jk})$ with $j, k \in \{0, \dots, 4\}$.

In our example we have $\Phi_{jk} \in \text{Div}_{(2)}(\mathcal{E})$, because the \mathbf{D}_{jk} are already orthogonal to all the other fibers. Writing Φ_{jk} as a linear combination of the components of the fiber above (2)

$$\Phi_{jk} = n_1 \mathcal{C}_1 + n_2 \mathcal{C}_2 + n_3 \mathcal{C}_3 + n_4 \mathcal{C}_4 + n_5 \mathcal{C}_5 + n_6 \mathcal{C}_6 + n_7 \mathcal{C}_7 + n_8 \mathcal{C}_8,$$

with $n_i \in \mathbb{Q}$ and solving the equations

$$(4.1) \quad (\mathbf{D}_{jk} + \Phi_{jk}, \mathcal{C}_l)_{fin} = 0, \quad 1 \leq l \leq 8$$

we get representatives for the coefficients n_i 's; observe Φ_{jk} is defined modulo \mathcal{E}_2 .

For Φ_{14} we calculate for instance

$$\Phi_{14} = -\frac{5}{4}\mathcal{C}_1 - \frac{3}{4}\mathcal{C}_2 - \frac{3}{2}\mathcal{C}_3 - \mathcal{C}_4 - \frac{1}{2}\mathcal{C}_5 + \frac{1}{2}\mathcal{C}_7.$$

Now we can calculate all the local intersection numbers that define the coefficients δ_p . Again, we will only calculate $(D_{14}, D_{32})_{fin}$ as an example, since all other intersection numbers are calculated in a similar way.

$$\begin{aligned} (D_{14}, D_{32})_{fin} &= (\mathbf{D}_{14} + \Phi_{14}, \mathbf{D}_{32})_{fin} \\ &= (\mathbf{S}_1 - \mathbf{S}_4 + \Phi_{14}, \mathbf{S}_3 - \mathbf{S}_2)_{fin} \\ &= (\mathbf{S}_1, \mathbf{S}_3)_{fin} - (\mathbf{S}_1, \mathbf{S}_2)_{fin} - (\mathbf{S}_4, \mathbf{S}_3)_{fin} \\ &\quad + (\mathbf{S}_4, \mathbf{S}_2)_{fin} + (\Phi_{14}, \mathbf{S}_3)_{fin} - (\Phi_{14}, \mathbf{S}_2)_{fin} \\ &= 0 - m_{12} \log(5) + m_{34} \log(5) \\ &\quad + m_{24} \log(2) + 0 + \frac{5}{4}(\mathcal{C}_1, \mathbf{D}_2)_{fin} \log(2) \\ &= 0 - \log(5) - \log(5) + 5 \log(2) + 0 + \frac{5}{4} \log(2) \\ &= \frac{25}{4} \log(2) - 2 \log(5) \end{aligned}$$

We can collect the intersection data into the following

Theorem 4.2. *Let $D_{ij} = S_i - S_j$ and $D_{kl} = S_k - S_l$ be two cuspidal divisors on (E, β_E) with no common component. We write $(D_{ij}, D_{kl})_{fin} = \sum_p \delta_p \log(p)$. Then only the δ_p for $p \in \{2, 5\}$ will be different from zero and they are given by the table*

D_{ij}	D_{kl}	δ_2	δ_5
$S_1 - S_4$	$S_3 - S_2$	6, 25	-2
$S_1 - S_3$	$S_4 - S_2$	0	-2
$S_1 - S_2$	$S_3 - S_4$	6, 25	0
$S_1 - S_4$	$S_3 - S_0$	1, 25	-1
$S_1 - S_3$	$S_4 - S_0$	0	-1
$S_1 - S_0$	$S_3 - S_4$	1, 25	0
$S_1 - S_4$	$S_0 - S_2$	5	-1
$S_1 - S_2$	$S_0 - S_4$	5	0

Proof: The necessary calculations to achieve the table have been explained above; a detailed calculation is in [Bu]. \square

Remark 4.3. *The information from the table in theorem 4.2 alone will never be sufficient to calculate values of scattering constants, since the value of the formula in theorem 2.5 stays unchanged when the scattering constants differ by a constant term: Take for example $D_{ij} = S_i - S_j$ and $D_{kl} = S_k - S_l$ then on the left hand side of theorem 2.5 we get the sum $C_{ik} - C_{il} - C_{jk} + C_{jl}$ which would equal $(C_{ik} + c) - (C_{il} + c) - (C_{jk} + c) + (C_{jl} + c)$.*

Remark 4.4. *Most of the calculations in this chapter can be done algorithmically, a detailed description of such a refined version of Tate's algorithm can be found in [Bu2].*

5. LINEAR RELATIONS FOR $C_{jk}^{\Gamma_E}$

We will use the properties of scattering constants introduced in chapter 1 to find identities of and linear dependencies between the scattering constants of the pair (E, β_E) to fix the constant mentioned in remark 4.3.

Proposition 5.1. *For the group Γ_E associated with the Belyi pair (E, β_E) it suffices to know the scattering constants C_{14}, C_{34}, C_{12} and $C^{\Gamma(1)}$. Then all scattering constants are known and we have the list:*

$$\begin{aligned}
C_{00} &= \frac{1}{30} \left(C^{\Gamma(1)} - \frac{1}{\pi} (14 \log(2) + 6 \log(5)) \right) \\
C_{01} &= \frac{1}{30} \left(C^{\Gamma(1)} - \frac{1}{\pi} (8 \log(2) + 3 \log(5)) \right) \\
C_{02} &= \frac{1}{30} \left(C^{\Gamma(1)} - \frac{1}{\pi} (2 \log(2) + 3 \log(5)) \right) \\
C_{03} &= \frac{1}{30} \left(C^{\Gamma(1)} - \frac{1}{\pi} (8 \log(2) + 3 \log(5)) \right) \\
C_{04} &= \frac{1}{30} \left(C^{\Gamma(1)} - \frac{1}{\pi} (2 \log(2) + 3 \log(5)) \right) \\
C_{11} &= \frac{1}{4} \left(\frac{1}{6} C^{\Gamma(1)} - \frac{62}{15\pi} \log(2) - C_{12} \right) \\
C_{12} & \\
C_{13} &= \frac{1}{4} \left(\frac{1}{6} C^{\Gamma(1)} - \frac{32}{15\pi} \log(2) - C_{14} \right) \\
C_{14} & \\
C_{22} &= \frac{1}{6} C^{\Gamma(1)} - \frac{47}{15\pi} \log(2) - 4C_{12} \\
C_{23} &= C_{14} \\
C_{24} &= \frac{1}{6} C^{\Gamma(1)} - \frac{17}{15\pi} \log(2) - 4C_{14} \\
C_{33} &= \frac{1}{4} \left(\frac{1}{6} C^{\Gamma(1)} - \frac{62}{15\pi} \log(2) - C_{34} \right) \\
C_{34} & \\
C_{44} &= \frac{1}{6} C^{\Gamma(1)} - \frac{47}{15\pi} \log(2) - 4C_{34}
\end{aligned}$$

Proof: First of all, remember, that the scattering matrix is symmetric, i.e. for any two cusps S and S' we have $C_{SS'} = C_{S'S}$. Hence, in the list above really all scattering constants occur.

Secondly, we apply proposition 1.3. Since the point S_0 is totally ramified the sum in formula (1.6) for $S_j = S_0$ is only to be taken over one single element. When we now differ S_k over all the cusps, we get the first 5 rows of the list above. For that we have to realize that the cusps have the following widths

$$b_0 = 10, \quad b_1 = 8, \quad b_2 = 2, \quad b_3 = 8, \quad b_4 = 2.$$

The widths are always two times the ramification index, since the widths of all the cusps of $\Gamma(2)$ are two.

Now we apply proposition 1.3 again but in the cases not involving S_0 . Then we get

$$4C_{11} + C_{12} = \frac{1}{6} C^{\Gamma(1)} - \frac{62}{15\pi} \log(2)$$

$$\begin{aligned}
4C_{33} + C_{34} &= \frac{1}{6}C^{\Gamma(1)} - \frac{62}{15\pi}\log(2) \\
4C_{13} + C_{14} &= \frac{1}{6}C^{\Gamma(1)} - \frac{32}{15\pi}\log(2) \\
4C_{13} + C_{23} &= \frac{1}{6}C^{\Gamma(1)} - \frac{32}{15\pi}\log(2) \\
4C_{41} + C_{42} &= \frac{1}{6}C^{\Gamma(1)} - \frac{17}{15\pi}\log(2) \\
4C_{23} + C_{24} &= \frac{1}{6}C^{\Gamma(1)} - \frac{17}{15\pi}\log(2) \\
4C_{43} + C_{44} &= \frac{1}{6}C^{\Gamma(1)} - \frac{47}{15\pi}\log(2) \\
4C_{12} + C_{22} &= \frac{1}{6}C^{\Gamma(1)} - \frac{47}{15\pi}\log(2).
\end{aligned}$$

This follows directly with the widths from above and the fact that cusps are equivalent under $\Gamma(2)$ if and only if they have the same image under β_E .

From this identities the list in the proposition follows. \square

Theorem 4.2 may now be used to calculate the missing scattering constants.

6. PROOF OF THE MAIN RESULT

We observe, that on an elliptic curve C the Néron Tate pairing on $\text{Div}_0(C)$ is compatible with the Néron Tate pairing on the Mordell Weil group, i.e. for $S, S' \in C(\mathbb{Q})$

$$\langle S - \mathcal{O}, S' - \mathcal{O} \rangle_{NT} = \langle S, S' \rangle_{NT}.$$

In particular, if $C(\mathbb{Q})$ is generated by one point P then the values of all Néron Tate pairings in $C(\mathbb{Q})$ are multiples of the Néron Tate height of the generator $\text{ht}_{NT}(P)$. Thus, by bilinearity and insertion of \mathcal{O} , for $D, D' \in \text{Div}_0(C(\mathbb{Q}))$ there is a $n \in \mathbb{Z}$ such that

$$\langle D, D' \rangle_{NT} = n \cdot \text{ht}_{NT}(P).$$

We can use this fact to simplify the left hand side of equation (2.1) in theorem 2.5.

For the elliptic curve E considered in this text we have in the Mordell Weil group $S_2 = -4S_1$, $S_3 = -S_1$, $S_4 = 4S_1$ and $S_0 = \mathcal{O}$ (one can use e.g. the computer algebra system pari to calculate multiples of S_1). Thus we get

$$\begin{aligned}
\langle S_1 - S_2, S_0 - S_4 \rangle_{NT} &= \langle S_1, S_0 \rangle_{NT} - \langle S_1, S_4 \rangle_{NT} - \langle S_2, S_0 \rangle_{NT} + \langle S_2, S_4 \rangle_{NT} \\
&= 0 - 4\text{ht}_{NT}(S_1) - 0 - 16\text{ht}_{NT}(S_1) \\
(6.1) \quad &= -20\text{ht}_{NT}(S_1).
\end{aligned}$$

Theorem 6.1. *For the group Γ_E associated with (E, β_E) we have*

$$\begin{aligned}
C_{14} &= \frac{1}{30} \left(C^{\Gamma(1)} + \frac{1}{\pi} (7\log(2) - 60\text{ht}_{NT}(S_1)) \right) \\
C_{21} &= \frac{1}{30} \left(C^{\Gamma(1)} + \frac{1}{\pi} (7\log(2) - 15\log(5) + 60\text{ht}_{NT}(S_1)) \right) \\
C_{34} &= \frac{1}{30} \left(C^{\Gamma(1)} + \frac{1}{\pi} (7\log(2) - 15\log(5) + 60\text{ht}_{NT}(S_1)) \right)
\end{aligned}$$

and $S_1 = (1, 4)$ is, like before, the generator of the Mordell Weil group $E(\mathbb{Q})$.

Proof: We insert information from theorem 4.2 into theorem 2.5. If we use the last row from the table in theorem 4.2, formula (2.1) becomes

$$(6.2) \quad \langle S_1 - S_2, S_0 - S_4 \rangle_{NT} = -5 \log(2) - 2\pi (C_{01} - C_{14} - C_{02} + C_{42}).$$

With the list from proposition 5.1 we simplify the sum of scattering constants to

$$(6.3) \quad 2\pi (C_{01} - C_{14} - C_{02} + C_{24}) = \frac{\pi}{3} C^{\Gamma(1)} - \frac{8}{3} \log(2) - 10\pi C_{14}.$$

As seen in (6.1) the right hand side of equation (6.2) becomes $-20 \text{ht}_{NT}(S_1)$. Hence, the first scattering constant is given by

$$(6.4) \quad C_{14} = \frac{1}{30} \left(C^{\Gamma(1)} + \frac{1}{\pi} (7 \log(2) - 60 \text{ht}_{NT}(S_1)) \right).$$

For the last scattering constants, we repeat the calculation above with taking the second to last and the fifth row (respectively) from the table in theorem 4.2 and the result for C_{14} into account and conclude the constants to be

$$(6.5) \quad C_{12} = C_{34} = \frac{1}{30} \left(C^{\Gamma(1)} + \frac{1}{\pi} (7 \log(2) - 15 \log(5) + 60 \text{ht}_{NT}(S_1)) \right).$$

□

If we insert these results into the formulas of proposition 5.1 we get a description of the scattering constants for Γ_E in $C^{\Gamma(1)}$, the log's of 2 and 5 and $\text{ht}_{NT}(S_1)$.

Remark 6.2. *Good numerical approximations exist for the Néron Tate height. By means of the computer algebra system pari we obtain*

$$\text{ht}_{NT}(S_1) \approx 0.1283750629460508690621759.$$

This leads to the following numerical approximations of scattering constants for Γ_E :

$C_{00} \approx -0.176518865559$	$C_{01} \approx -0.0811617456560$
$C_{02} \approx -0.0370346256255$	$C_{03} \approx -0.0811617456560$
$C_{04} \approx -0.0370346256255$	$C_{11} \approx -0.168350141906$
$C_{12} \approx -0.0940378417200$	$C_{13} \approx -0.0812067899954$
$C_{14} \approx -0.00134004905741$	$C_{22} \approx -0.170651442311$
$C_{23} \approx -0.00134004905741$	$C_{24} \approx -0.100171412657$
$C_{33} \approx -0.168350141906$	$C_{34} \approx -0.0940378417200$
$C_{44} \approx -0.170651442311$	

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